Exponents and Logarithms

Exponents:

Exponents, also known as powers, are a mathematical notation that indicates the number of times a number is multiplied by itself. They are written as a small number to the right and above the base number.

Example:

$$5^3 = 5 \cdot 5 \cdot 5 = 125$$

Rule 1: Multiplying exponents

Expressions where exponents are being multiplied can only be simplified if the base is the same otherwise the expression cannot be simplified unless you convert the exponents to their natural form and then multiply them. When multiplying exponents with the same base you should add the powers together. Here are some examples:

$$3^5 \cdot 3^2 = 3^{(5+2)} = 3^7$$

$$2^5 \cdot 2 = 2^{(5+1)} = 2^6$$

$$2^5 \cdot 2^{-3} = 2^{(5-3)} = 2^2$$

However, if the base is not the same, they should be kept as it is or they can be converted to their natural form and then multiplied. Here is an example:

$$2^2 \cdot 3^6 = 4 \cdot 729 = 2916$$

In cases where there are variables present in the exponential expression and you do not know the value of the variables you should leave the expression as it is.

Rule 2: Dividing exponents

When dividing exponents, you subtract the powers as long as the base of the exponents are the same; otherwise, you keep it the same. Here are some examples:

$$\frac{5^3}{5} = 5^{(3-1)} = 5^2$$

$$\frac{4^{10}}{4^2} = 4^{(10-2)} = 4^8$$

$$\frac{2^{-5}}{2^3} = 2^{(-5-3)} = 2^{-8}$$

$$\frac{5^3}{2^5} \rightarrow Cannot\ be\ simplified\ to\ a\ power \rightarrow \frac{125}{32} = 3.90625$$

Rule 3: Multiplying exponents in brackets

When there is an exponent outside the bracket, to simplify the expression you must multiply the exponent values to all the powers inside the bracket regardless of the base. Here are some examples:

$$\left(5^4\right)^2 = 5^{(4\cdot 2)} = 5^8$$

$$\left(2x^2\right)^3 = \left(2^{(1\cdot 3)}x^{(2\cdot 3)}\right) = 2^3x^6 = 8x^6$$

$$\left(x^{-2}y\right)^4 = x^{(-2\cdot 4)}y^{(4\cdot 1)} = x^{-8}y^4$$

Rule 4: Negative exponents

When there is a negative exponent, you must take the reciprocal of the base and then add the positive power of the original power to the base value to simplify the expression. Here are some examples:

$$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$$

$$4^{-4} = \frac{1}{4^4} = \frac{1}{256}$$

$$xy^{-3} = \frac{x}{y^3}$$

$$\left(2x^2y^{-2}\right)^2 = 4x^4y^{-4} = \frac{4x^4}{y^4}$$

Rule 5: Fraction exponents

When there is a base taken to the power of a fraction, you must consider the denominator of the fraction as the power of the root you will take from the base. Also, the numerator of the fraction will be the power of the base inside the root. Here are some examples:

$$5^{\frac{2}{3}} = \sqrt[3]{5^2} = 2.92$$

$$x^{\frac{4}{3}} = \sqrt[3]{x^4}$$

$$\left(x^3y\right)^{\frac{1}{2}} = x^{\frac{3}{2}}y^{\frac{1}{2}} = \left(\sqrt{x^3}\right)\left(\sqrt{y}\right) = \sqrt{x^3y}$$

$$25^{\frac{1}{2}} = \sqrt{25} = 5$$

Examples of a mix of all the rules:

$$\left(3x^{\frac{1}{2}}y\right)^{-\frac{3}{2}} = \left(3\sqrt{x}\,y\right)^{-\frac{3}{2}} = \left(\frac{1}{\left(3\sqrt{x}\,y\right)}\right)^{\frac{3}{2}} = \frac{1}{\left(\sqrt{3^3}\cdot\sqrt[4]{x^3}\cdot\sqrt{y^3}\right)}$$

$$25\cdot 125 = 5^2\cdot 5^3 = 5^{(2+3)} = 5^5 = 3125$$

$$(4x)^{-\frac{1}{2}}\cdot\left(\frac{1}{2}\right)^{-1} = \frac{1}{(4x)^{\frac{1}{2}}}\cdot 2 = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$$

Logarithms:

Logarithms are the inverse operations of exponentiation. In mathematics, a logarithm of a number is the exponent to which another fixed number, called the base, must be raised to produce that number.

Example:

$$\log_a(y) = x \rightarrow a^x = y$$

$$5^6 = 15625 \rightarrow \log_5(15625) = 6$$

$$\log_4(64) = x \rightarrow 4^x = 64 \rightarrow x = 3$$

$$4^y = 256 \rightarrow \log_4(256) = 4 \rightarrow y = 4$$

Basic form:

If the logarithm is present without an apparent base, it is mathematically assumed that the logarithm has a base of 10. Also, if a logarithm has a base of e, it can be re-written as ln.

The rules of logarithms are the same rules which will apply to ln as it can be re-written as logarithm base e anyways. Here are some examples:

$$\log(a) = \log_{10}(a)$$

$$\ln(x) = \log_{e}(x)$$

<u>Rule 1: Adding logarithms</u>

When adding two logarithms they must have the same base in order for this rule to be applicable. Once the two bases are the same you just multiply the two values from the parenthesis and simplify the expression into a singular logarithmic statement. Here are some examples:

$$\log_a(b) + \log_a(y) = \log_a(b \cdot y) = \log_a(by)$$

$$\log_5(50) + \log_5(6) = \log_5(300)$$

$$\log_3(x) + \log_3(4) = \log_3(4x)$$

$$\log_5(3) + \log_3(4) \rightarrow \textit{cannot be done because base is not same}$$

<u>Rule 2: Subtracting logarithms</u>

When subtracting logarithms, like adding they must have the same base. Once they have the same base you can divide the two values being subtracted from the two different parenthesis and simplify the logarithmic expression into a singular logarithmic statement. Here are some examples:

$$\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$$

$$\log_3(15) - \log_3(5) = \log_3\left(\frac{15}{5}\right) = \log_3(3) = 1$$

$$\log_x(4) - \log_x(y) = \log_x\left(\frac{4}{y}\right)$$

$$\log_3(5) - \log_5(3) \rightarrow \textit{cannot be done because base is not same}$$

Rule 3: Power of the logarithm

When there is a power to the base of the value inside the parenthesis of a logarithm, that power can be taken out as the coefficient of the logarithm instead. Here are some examples:

$$k \cdot \log_a(x) = \log_a\left(x^k\right)$$

$$\log_3\left(4^2\right) = 2\log_3(4)$$

$$4\log_5(6) = \log_5\left(6^4\right)$$

$$2\log_3(3) = \log_3\left(3^2\right) = 2$$

Rule 4: Base to the power of a logarithm

When there is a base raised to the power of a logarithm, the answer is the same value as the value in the parenthesis of the logarithm. The base of the logarithm has to be the same as the base of the exponent. Here are some examples:

$$b^{\log_b(k)} = k$$

$$5^{\log_5(60)} = 60$$

$$3^{\log_3(x)} = x$$

<u>Rule 5: Changing the base of a logarithm</u>

When you have two logarithmic expressions present which you want to simplify or operate with, you can do so by changing the base of each logarithm. You can divide a logarithm base c with the value of the original parenthesis by logarithm with the same new base and parenthesis of the original base. Here are some examples:

$$\log_b(a) = \frac{\log_c(b)}{\log_c(a)}$$

$$\log_3(81) = \frac{\log_4(81)}{\log_4(3)}$$

$$\log_9(81) = \frac{\log_3(243)}{\log_3(9)} = \frac{5}{2} = 1.5$$

<u>Rule 6: Expanding the brackets</u>

When you have an expression in the power section of the value in the parenthesis, you can take that out as a bracket getting multiplied to the logarithm. Then you can expand the brackets. Here are some examples:

$$\log_x\left(a^{(c+y)}\right) = (c+y)\left(\log_x(a)\right) = c\log_x(a) + y\log_x(a)$$

$$\log_3\left(4^{(3+y)}\right) = (3+y)\left(\log_3(4)\right) = 3\log_3(4) + y\log_3(4)$$

Examples of complex equations:

$$e^{(3x+1)} = 5 \rightarrow \log_e(5) = 3x + 1 \rightarrow \ln(5) = 3x + 1 \rightarrow \frac{(\ln(5)-1)}{3} = x$$

$$\ln(3x+1) = 5 \rightarrow e^5 = 3x + 1 \rightarrow \frac{(e^5 - 1)}{3} = x = 49.1$$

$$\log_2(3x+1) = 5 \rightarrow 2^5 = 3x + 1 \rightarrow \frac{(2^5 - 1)}{3} = x = 10.33$$